MMP Learning Seminar

Week 91.

Content:

- Bounding length of blow-ups &
- Bounding multiplicities of lc places.

Log canonical thresholds of bounded degree.

Theorem 1.8: Let d, r be natural numbers and $\varepsilon > 0$.
There exists $t := t(d, r, \varepsilon) > 0$ satisfies the following. Assume:

- (X, B) is projective ε-lc of dim d,
- A very ample on X with $A^d \leq r$, $\leftarrow$ X is bounded
- $A - B$ is pseudo-effective, and $\leftarrow$ control sing (X, B).
- $M \geq 0$ $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, $A - M$ is pseff. Then

$$\operatorname{lct}(X, B, |M|_{\mathbb{R}}) \geqq \operatorname{lct}(X, B, |A|_{\mathbb{R}}) \geqq t.$$

Remark: In the previous statement supp B & supp M are not controlled.

Noether normalization: $X \xrightarrow{f} \mathbb{P}^n$ where $n = \dim X$.

finite surjective

(most of the time not Galois).

Alexander.

M a n-manifold then there exists a branched cover $M \longrightarrow S^n$
which branches along a smooth submanifold of dim $\leq n-2$.

in dim $= 2$, we branch along points.

in dim $= 3$, we branch along Borromean circles.

Proposition 5.2 (Finite morphisms to $\mathbb{P}^d$):

Let $(X, \Lambda = \sum_{i=1}^{d} S_i)$ be a log smooth projective pair of dim d. where Λ is reduced. $B = \sum_1^r b_j B_j \geq 0$ be an $\mathbb{R}$-divisor. Assume:

- $x \in \bigcap_{i=1}^{d} S_i$,
- Supp B does not contain strata of (X, Λ) except possibly x,
- A & $A - S_i$ are very ample.

There exists $\pi : X \longrightarrow \mathbb{P}^d$ finite surjective, $\pi(x) = z = [1:0:\cdots:0]$,

$\pi(S_i) = H_i$, π is étale around z, $\operatorname{supp}(B) \cap \pi^{-1}(z) \subseteq x$,

$\deg \pi = A^d$ and $\deg_{H_i}(C) \leq \deg_A B$ where $C = \sum_1^r b_j \pi(B_j)$.

Proof: $\alpha_0, \ldots, \alpha_d$ global sections of $\mathcal{O}_X(A)$

$X \xrightarrow{\pi} \mathbb{P}^d$ finite surjective

$\square$

Fix d, r, ε.

Proposition 5.5 (Bound on number of blow-ups): Assume

- (X,B) projective ε-lc d-dim,
- A very ample $A^d \leq r$,
- (X,Δ) log smooth Δ reduced, $\Delta \geq 0$.
- $\deg_A B \leq r$ and $\deg_A \Delta \leq r$,
- x is zero-dim strata of (X,Δ)
- $\operatorname{supp} B$ does not contain strata (X,Δ) except possibly x
- T is a log canonical place of (X,Δ) with center x.
- $a(T; X, B) \leq 1$.

$$\Theta = \sum_{i=1}^{d} H_i$$

$[1:0:\dots:0]$.

Then, T can be extracted by toroidal blow-ups wrt (X,Δ) at most $p = p(d,r,\varepsilon)$.

Sketch:

$$\begin{array}{ccc} (X, B, \Delta) & \overset{\text{replace}}{\rightsquigarrow} & (\mathbb{P}^d, C, \Theta) \\ \uparrow & & \uparrow \\ T & & R \end{array}$$

We do not know if $(\mathbb{P}^d, C)$ is ε-lc away from $z = [1:0:\dots:0]$.

Aim: modify C so that it is ε'-lc for some ε' only depending on d, r, ε.

Proof: **Step 3:** $(\mathbb{P}^d, \Theta + tC)$ is lc away from z

for some $t \in (0, 1/2)$ only depending $\boxed{t := t(d, r)}$

$(\mathbb{P}^d, \Theta + tC)$ will be klt away from $\operatorname{supp}(\Theta)$.

$y \in \operatorname{Supp} \Theta$, $y \neq z$. G minimal linear subs which a strata of Θ and contains y. G is positive-dimensional.

$\deg_{H'} C|_G \leq r$, H' is a hyperplane in G.

G is not in the support of C.

Inductively $(G, tC|_G)$ is klt for some $\boxed{t := t(d, r).}$

Inversion of adjunction implies $(\mathbb{P}^d, \Theta + tC)$ is lc in a ngbhd of y.

Step 4: We construct $\underline{\Delta}$ such that $(\mathbb{P}^d, \Delta)$ is ε'-lc,

$$\varepsilon' = \frac{t}{2}\varepsilon, \quad K_{\mathbb{P}^d} + \Delta \sim_{\mathbb{R}} 0 \quad \text{and} \quad \alpha(R, \mathbb{P}^d, \Delta) \leq 1.$$

Consider $D = (1 - t/2)\,\Theta + \frac{t}{2} C$. The pair $(\mathbb{P}^d, D)$ is ε'-lc.

If the center contains z, then $(\mathbb{P}^d, C)$ is ε-lc around it.

If the center does not contain z, this is not a lcc of $(\mathbb{P}^d, \Theta)$.

$$-(K_{\mathbb{P}^d} + D) = -\left(1 - \frac{t}{2}\right)\underbrace{(K_{\mathbb{P}^d} + \Theta)}_{\deg = 1} - \frac{t}{2}\underbrace{(K_{\mathbb{P}^d} + C)}_{\deg \geq d+1-r}$$

Taking t small enough, we can make sure $-(K_{\mathbb{P}^d} + D)$ ample.

By construction $\alpha(R, \mathbb{P}^d, D) \leq 1$

Take $0 \leq G \sim_{\mathbb{R}} -(K_{\mathbb{P}^d} + D)$ very general &

set $\Delta = D + G$.

Review:

$$
\begin{tikzcd}
 & & \Delta \arrow[r, squiggly] & \Omega \\
T \arrow[d] & R \arrow[d] & D \arrow[u, squiggly] & \\
(X, \Lambda, B) \arrow[r, squiggly] & (\mathbb{P}^d, \Theta, C) \arrow[ur, squiggly] & &
\end{tikzcd}
$$

$(\mathbb{P}^d, C)$ does not have nice sing away from Z.

$(\mathbb{P}^d, \Theta + tC)$ still lc away from Z.

Construct Δ such that $(\mathbb{P}^d, \Delta)$ is log CY,

ε'-lc & $\alpha(R, \mathbb{P}^d, \Delta) \leq 1$.

Remark: The coefficients of Δ are not controlled.

Step 5: $W' \xrightarrow{\phi} \mathbb{P}^d$ extracts R

$K_{W'} + \Delta_{W'}$ is the pull-back of $K_{\mathbb{P}^d} + \Delta$.

ε'-lc

$-K_{W'}$ is big. Run MMP on $-K_{W'}$.

$W' \dashrightarrow W''$ $\quad -K_{W''}$ is big & semiample

$W'' \longrightarrow W'''$ $\quad -K_{W'''}$ is ample.

$$(W', \Delta_{W'}) \dashrightarrow (W''', \Delta_{W'''}) \qquad \text{is } \varepsilon'\text{-lc}$$

We conclude that W''' is Fano, toric & ε'-lc.

Then W''' belongs to a bounded family.

It admits a klt n-complement.

We can push-forward this complement to $\mathbb{P}^d$.

$\underset{\text{klt}}{(\mathbb{P}^d, \Omega)}$ log CY, $n(K_{\mathbb{P}^d} + \Omega) \sim 0$, and $\alpha(R, \mathbb{P}^d, \Omega) \leq 1$.

Step 6: $(\mathbb{P}^d, \operatorname{supp}(\Omega + \Theta))$ is log bounded.

$(\mathbb{P}^d, \Omega + u\Theta)$ is lc for some u only depending on d, r, ε.

We conclude $\mu_R \phi^* \Theta \leq \frac{1}{u}$.

The length of blow-ups is at most $\lfloor \frac{1}{u} - 1 \rfloor$.

$\square$.

Proposition 5.7 (Bound of multiplicities at lcp):

Assume Thm 1.8 in $\dim \leq d-1$.

There exists $q := q(d, r, \underline{\underline{n}}, \epsilon)$ satisfying the following. Assume:

- (X, B) projective ϵ-lc d-dim,
- A very ample $A^d \leq r$,
- $\Delta \geq 0$ & $\underline{\underline{n}}\Delta$ integral.
- $L \geq 0$ is an $\mathbb{R}$-divisor,
- the divisors $A-B$, $A-\Delta$, & $A-L$ are pseudo-effective
- (X, Δ) is lc near x,
- T is a log canonical place of (X, Δ) with center x.
- $a(T; X, B) \leq 1$.

$\boxed{\text{If } \Delta \text{ is reduced}}$ + supp B does not contain strata of (X, Δ) other than x

For any resolution $\nu : U \longrightarrow X$ so that T is a divisor on U, we have that $\mu_T \nu^* L \leq q$.

Proof: Cutting with hyperplanes, we may assume x is closed.

We will show the statement when (X, Δ) is log smooth & Δ is reduced.

$\left\{ \textbf{Aim:} \text{ modify } B \text{ s.t its support does not contain strata of } (X, \Delta) \text{ that is not } x. \right\}$

(X, Δ) is log bounded $\quad (X', \Delta') \longrightarrow (X, \Delta)$

Step 1: We may assume x is the only 0-strata of (X, Δ)

Step 2: We show that $(X, (1+t)B)$ is $\frac{\varepsilon}{2}$-lc

away from finitely many points. $t := t(d, n, r, \varepsilon)$.

$H \in |A|$ general element.

By Thm 1.8 in dim $d-1$ we know that

$(H, B_H + at B_H)$ is klt

By inversion of adjunction $(X, B + atB)$ is klt near H.

$$\Downarrow$$

$(X, B + atB)$ is klt outside finitely many points

We conclude, taking average that $(X, (1+t)B)$ is $\frac{\varepsilon}{2}$-lc

Step 3: $\psi : V \to X$ log resolution.
$\quad\; \cup\!\!\mid$
$\quad\; T$

log discrepancies wrt (X, B)

$$\Gamma_V = (1+t)\widetilde{B} + \left(1 - \tfrac{\varepsilon}{4}\right) \sum_i' E_i + (1-a)T$$

$$K_V + \Gamma_V = \psi^*(K_X + B) + t\widetilde{B} + F \quad \text{where} \quad F = \sum_i' \left(a_i - \tfrac{\varepsilon}{4}\right) E_i .$$

$$K_V + \Gamma_V = \psi^*(K_X + (1+t)B) + G \quad \text{where} \quad G = \sum_i' \left(a_i' - \tfrac{\varepsilon}{4}\right) E_i + (a' - a)T$$

log discrepancies wrt $(X, (1+t)B)$.

Step 4: Run a $(K_V + \Gamma_V)$-MMP over X.

contract all divisors in the support of G with coeff > 0

$K_V + \Gamma_V \equiv t\widetilde{B} + F / X$, so T is not contracted by this MMP.

$\pi : Y \to X$ that extracts T and is an isom away from finitely points.

Step 5: We construct $0 \leq D_Y$ on Y.

$A_Y = \pi^* A$,

$K_Y + \Gamma_Y + 3d A_Y$ semiample & big.

$0 \leq D_Y \sim_{\mathbb{R}} \frac{1}{t}(K_Y + \Gamma_Y + 3d A_Y)$ with coeff $\leq 1-\varepsilon$.

$D_Y = \pi^* H + \widetilde{B}_Y + \frac{1}{t} F_Y$, for some

$H \sim_{\mathbb{R}} \frac{1}{t}(K_X + B + 3d A)$.

Denote by D the push-forward of D_Y to X.

By construction $D = H + B$.

Step 6: (X, B) is ε-lc

$(Y, \widetilde{B} + R_Y)$ sub-ε-lc.

$(Y, R_Y + D_Y - \frac{1}{t} F_Y)$ is sub-ε-lc

very general divisor

$$\| \\ \pi^*(X, D)$$

(X, D) is ε-lc.

$a(T, X, D) \leq 1$.

$3mA - D$ is ample

We can replace $B \rightsquigarrow D$.

& D does not contain strata of (X, Δ) diff than x.

$\square$

$\underbrace{K_Y + \Gamma_Y}_{\text{klt}} + 3d A_Y$ semiample & big.

$A_Y . C \geq 0$ Cartier.

$A_Y . C \geq 1$

$3d A_Y . C \geq 3d.$

then the extremal negative rays are spanned by curves for which $(K_Y + \Gamma_Y) . C > -2 \dim Y.$

Boundedness of extremal lengths.